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The propagation of gravitational radiation in dense media

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Abstract. We solve Einstein's field equations for gravitational radiation propagating in a background metric given by the interior Schwarzschild solution. Expressions for the transmitted fraction as a function of frequency and propagation distance are found. Absorption is found to be present only for frequencies less than a critical frequency, the 'gravitational plasma frequency', which is roughly proportional to the square root of the mean density of the medium. The plasma frequency ν_{crit} is found to be a useful concept for gravitational radiation, though the fraction transmitted does not fall as rapidly with distance of propagation below ν_{crit} as it does in the analogous electromagnetic case and the transition to total transmission is not as abrupt as a function of frequency. For a neutron star $\nu_{crit} = 6700$ Hz, well above the low frequencies of most current gravitational radiation detectors.

1. Introduction

Considerable attention has been given in recent years to the production and detection of gravitational radiation (many references are given in Press and Thorne 1972 and Misner *et al* 1973) occasioned by the apparent discovery of such radiation by Weber (1969, 1970a, b). Attention has also been given to the propagation of gravitational radiation with its refraction redshift, and backscatter. Little or no work has been done however on a possible 'gravitational plasma frequency'. In studying gravitational radiation propagating in the Robertson–Walker background metric, Weinberg (1972) found absorption at low frequencies and plane wave type propagation at high frequencies with a very gradual transition between the two kinds of behaviour. For this situation a 'gravitational plasma frequency' seems to be meaningless. In this paper we want to explore the propagation of gravitational radiation inside dense stars as a function of frequency and look in particular to see if a 'gravitational plasma frequency' characterizes the propagation. Below we consider a background metric given by the interior Schwarzschild solution and find a relatively sharp transition between absorption and free propagation with a meaningful gravitational plasma frequency. For a density of $\rho = 10^{14}\ \mathrm{g\ cm^{-3}}$ the critical frequency found below is 6700 Hz. Calculations, for example, of gravitational radiation from a vibrating neutron star which neglect absorption at frequencies below this will be in error.

2. Solution of field equations

We want to find radiative solutions to Einstein's field equations corresponding to gravitational waves propagating through dense media. Specifically let us consider weak gravitational radiation in a background metric given by the interior Schwarzschild solution for an incompressible fluid, and investigate its absorption as it travels through the star. The metric is $g_{\mu\nu}+h_{\mu\nu}$ where $h_{\mu\nu}$ is small and $g_{\mu\nu}$ is given by the line element of the interior Schwarzschild metric in isotropic coordinates from Møller (1952) as

$$ds^2 = V\,d\sigma^2 - Wc^2\,dt^2 \tag{1}$$

where

$$V \equiv 4h^2R^2(h^2+r^2)^{-2} \tag{2}$$

$$W \equiv [A-\tfrac{1}{2}(h^2-r^2)(h^2+r^2)^{-1}]^2 \tag{3}$$

$$A \equiv \tfrac{3}{2}(1-r_s'^2/R^2)^{1/2} \tag{4}$$

$$R^2 \equiv 3c^2/8\pi G\rho = r_s'^3/2m \tag{5}$$

$$h^2 \equiv r_s^2(A+\tfrac{1}{2}-\gamma)(\tfrac{1}{2}+\gamma-A)^{-1} \tag{6}$$

$$\gamma \equiv (1-m/2r_s)(1+m/2r_s)^{-1} \tag{7}$$

$$r_s \equiv \frac{hr_s'/R}{1+(1-r_s'^2/R^2)^{1/2}}. \tag{8}$$

In these definitions ρ is the average density of matter in the star, $m \equiv GM/c^2$, M is the mass of the star, G is the gravitational constant and r_s' is the radius of the star defined through $M \equiv (4\pi/3)r_s'^3\rho$; r_s is the radius of the star in the isotropic coordinates we are using. It is not related to M and ρ simply and is given in terms of r_s' by (8) which also relates standard and isotropic coordinates in general, if the subscripts are dropped.

We will work in the transverse traceless (TT) gauge of Misner *et al* (1973) for $h_{\mu\nu}$ corresponding to radiative solutions. A radiative solution can always be put into TT gauge whereas non-radiative $h_{\mu\nu}$ cannot. The coordinate conditions are then

$$\begin{aligned} &h_{\mu 0} = 0 \\ &h_{kj|j} = 0 \quad \text{(summed)} \\ &h_{kk} = 0 \quad \text{(summed)}, \end{aligned} \tag{9}$$

where a single vertical bar denotes ordinary differentiation and a double vertical bar denotes covariant differentiation. Our final results will be gauge invariant as we will see later. The perturbation $h_{\mu\nu}$ produces in the Ricci tensor a perturbation

$$\delta R_{\mu\nu} = (\delta\Gamma_{\alpha}{}^{\alpha}{}_{\mu})_{\|\nu} - (\delta\Gamma_{\mu}{}^{\alpha}{}_{\nu})_{\|\alpha} \tag{10}$$

where the perturbations in the affine connections are given by

$$\delta\Gamma_{\nu}{}^{\mu}{}_{\alpha} = \tfrac{1}{2}g^{\mu\rho}(h_{\rho\nu\|\alpha}+h_{\rho\alpha\|\nu}-h_{\nu\alpha\|\rho}). \tag{11}$$

Using the coordinate conditions (9) and the metric (1) gives

$$\delta R_{00} = 0 \tag{12}$$

$$\delta R_{0i} = \frac{1}{2V}X_j h_{ij|0}\left(\frac{V^\dagger}{V}+\frac{W^\dagger}{W}\right) \tag{13}$$

$$\delta R_{jk} = \text{complicated expression}, \tag{14}$$

where a dagger † denotes $d/d(r^2)$. Now the unperturbed energy momentum tensor is

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^{\mu} U^{\nu} + \frac{P}{c^2} g^{\mu\nu} \tag{15}$$

where P is the pressure. If we define

$$S_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha} \tag{16}$$

then the perturbations in $S_{\mu\nu}$ due to $h_{\mu\nu}$ are given by

$$\delta S_{ij} = \frac{1}{2}\left(\rho - \frac{P}{c^2}\right) h_{ij} \tag{17}$$

$$\delta S_{\mu 0} = 0 \tag{18}$$

where the first-order changes in ρ, P and $T^{\alpha}{}_{\alpha}$ are set equal to zero for radiative solutions from Weinberg (1972). The Einstein equations

$$\delta R_{\mu\nu} = -(8\pi G/c^2)\delta S_{\mu\nu} \tag{19}$$

are satisfied identically for $\mu = 0$, $\nu = 0$. From (13) and (18) we must have the additional restriction that $X_j h_{ij|0} = 0$ for (19) to hold for $\mu = 0$, $\nu = i$. We shall limit ourselves to the class of solutions for which $X_j h_{ij} = 0$. With this restriction (14) greatly simplifies and we have

$$\begin{aligned} \delta R_{jk} \equiv & -\frac{1}{2W} h_{jk|0|0} + \frac{1}{2V} h_{jk|i|i} + X_i h_{jk|i}\left(\frac{W^{\dagger}}{2WV} - \frac{5V^{\dagger}}{2V^2}\right) + h_{jk}\left(\frac{2V^{\dagger}}{V^2} + \frac{2V^{\dagger 2}}{V^3} r^2\right) \\ & = -\frac{4\pi G}{c^2}\left(\rho - \frac{P}{c^2}\right) h_{jk} \end{aligned} \tag{20}$$

where we used (17). This is the equation we must solve for h_{jk}. In flat space $\rho = 0$, $P = 0$, $V = 1$, $W = 1$ so that (20) reduces to $\Box^2 h_{jk} = 0$ with plane wave solutions as it must. We can simplify (20) if we note that for all systems of physical interest $h^2 \gg r^2$ for $r < r_s$. A good approximation is then to neglect r^2 relative to h^2 in (2) and (3) and in $V^{\dagger}$ and $W^{\dagger}$. Let us also take the time dependence of h_{jk} to be of the form $h_{jk} \propto e^{i\omega t}$. Then (20) can be written as

$$\nabla^2 h_{jk} + I X_i h_{jk|i} + J h_{jk} = 0 \tag{21}$$

where

$$I \simeq \frac{1}{h^2}\left(\frac{2}{A - \frac{1}{2}} + 10\right) \tag{22}$$

and

$$J \simeq \frac{4}{h^2}\left(\frac{\omega^2 R^2}{(A - \frac{1}{2})^2 c^2} + 1 - \frac{3P}{\rho c^2}\right). \tag{23}$$

J contains the right-hand side of (20). For a neutron star of one solar mass (the least favourable case for the above approximations), $\rho = 10^{14}\ \mathrm{g\,cm^{-3}}$, $h^2 = 20{\cdot}6 r_s^2$, $r_s = 15{\cdot}2$ km, $R = 40$ km, $r_s' = 16{\cdot}8$ km and $A = 1{\cdot}36$ so that our approximation $h^2 \gg r^2$ for $r < r_s$ is still quite good. In terms of m and r_s' this approximation implies that we must have

$$[1 + (1 - 2m/r_s')^{1/2}]^2 \gg 2m/r_s'$$

from (8) and (5). Thus for densities in excess of 10^{14} g cm^{-3} this approximation breaks down. Our attention will be restricted to $\rho \leqslant 10^{14}$ g cm^{-3} below.

The only r dependence in J now comes in through the pressure term. From Møller (1952) this pressure term can be written as

$$\frac{3P}{\rho c^2} = \frac{\frac{3}{2}(1-r'^2/R^2)^{1/2}-A}{R^2[A-\frac{1}{2}(1-r'^2/R^2)^{1/2}]} \tag{24}$$

where I have written this in terms of the radial coordinate r' instead of the isotropic coordinate r because the pressure and the constant A are both much simpler in terms of r'; r and r' are related by (8). P vanishes at the star's surface and is maximum at the centre.

$$\left.\frac{3P}{\rho c^2}\right|_{\max} = \frac{\frac{3}{2}-A}{R^2(A-\frac{1}{2})} = 0{\cdot}161 \tag{25}$$

in the least favourable case when $\rho = 10^{14}$. In the following we shall neglect the $3P/\rho c^2$ term in (23) so that

$$J \simeq \frac{4}{h^2}\left(\frac{\omega^2 R^2}{(A-\frac{1}{2})^2 c^2}+1\right) \tag{26}$$

and both J and I are constants independent of r. This is an excellent approximation for high frequencies where $\omega^2 R^2/c^2 \gg 1$. As $\omega R \to 0$ errors of the order of 15% will be introduced which will not materially affect either our conclusions that follow or their physical applicability.

Equation (21) is now tractable and can be separated easily and solved as it stands. Let us now restrict ourselves to the case of waves travelling in the X direction. No significant aspect of the problem is lost in doing this and it simplifies things somewhat. Equation (21) then becomes

$$\frac{\mathrm{d}^2 h_{jk}}{\mathrm{d}X^2}+IX\frac{\mathrm{d}h_{jk}}{\mathrm{d}X}+Jh_{jk} = 0. \tag{27}$$

This is the differential equation we must solve. Note that the quantity A is real in (22) and (23) if and only if $r_s'^2/R^2 \equiv 2m/r_s' \leqslant 1$. This is just the condition that the star is not collapsed to a black hole.

We can write (27) in terms of the dimensionless variable $z \equiv X\sqrt{(I/2)}$ as

$$\frac{\mathrm{d}^2 h_{jk}}{\mathrm{d}z^2}+2z\frac{\mathrm{d}h_{jk}}{\mathrm{d}z}+2\frac{J}{I}h_{jk} = 0. \tag{28}$$

Let $h_{jk} = Q\,\mathrm{e}^{-z^2}$ where we suppress the indices on Q_{jk} for convenience. The differential equation for Q is then

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q-2z\frac{\mathrm{d}}{\mathrm{d}z}Q+2\delta Q = 0 \tag{29}$$

where

$$\delta \equiv \frac{J}{I}-1 = 2\left(\frac{\omega^2 R^2}{(A-\frac{1}{2})^2 c^2}+1\right)\left(\frac{1}{A-\frac{1}{2}}+5\right)^{-1}-1.$$

For stars with densities $\rho < 10^{12}$, $\delta \simeq \frac{1}{3}(\omega^2 R^2/c^2-2)$. This is fairly accurate even at $\rho = 10^{14}$. The variable $z \leqslant 0{\cdot}6$ inside stars with $\rho \lesssim 10^{14}$ g cm^{-3}.

One solution of (29) for integer δ is

$$Q_1(z) = C_\delta H_\delta(z) \tag{30}$$

where C_δ is a constant and H_δ is the Hermite polynomial of order δ from Abramowitz and Stegun (1964). $H_\delta(z)$ obeys the recurrence relation

$$H_{\delta+1} = 2zH_\delta - 2\delta H_{\delta-1} \tag{31}$$

and we also have

$$\frac{\mathrm{d}H_\delta}{\mathrm{d}z} = 2\delta H_{\delta-1}. \tag{32}$$

For the recursion relation $H_0 \equiv 1$ and $H_1 = 2z$. A second linearly independent solution to our second-order differential equation (29) for integer δ is

$$Q_2(z) = D_\delta P_\delta(z) \tag{33}$$

where D_δ is a constant and

$$D_\delta P_\delta(z) = C_\delta H_\delta(z) \int_0^z \frac{\mathrm{e}^{t^2}\,\mathrm{d}t}{C_\delta{}^2 H_\delta{}^2(t)}. \tag{34}$$

The $P_\delta(z)$ defined by (34) satisfy the same relations (31) and (32) as $H_\delta(z)$. We can calculate them using the same recursion relation (31) but with

$$P_0(z) = \int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t$$

and

$$P_1(z) = -\mathrm{e}^{z^2} + 2z \int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t. \tag{35}$$

The $P_\delta(z)$ are not polynomials but are very closely related to the $H_\delta(z)$ in much the same way that $\cos kX$ is related to $\sin kX$.

For very small z we expect the solution of (28) to be

$$h_{jk}(z) = \mathrm{Re}\, e_{jk} \exp\{\mathrm{i}[(\delta+1)2]^{1/2}z\} + \mathrm{Re}\, e_{jk}^* \exp\{-\mathrm{i}[(\delta+1)2]^{1/2}z\} \tag{36}$$

where e_{jk} is a unit polarization tensor with components restricted by the TT coordinate conditions. We can normalize our solutions (30) and (33) defined by the recurrence relations by requiring that

$$C_\delta\, \mathrm{e}^{-z^2} H_\delta(z) \to \begin{cases} \cos[2(\delta+1)]^{1/2}z & \delta \text{ even} \\ \sin[2(\delta+1)]^{1/2}z & \delta \text{ odd} \end{cases} \tag{37}$$

and

$$D_\delta\, \mathrm{e}^{-z^2} P_\delta(z) \to \begin{cases} \sin[2(\delta+1)]^{1/2}z & \delta \text{ even} \\ \cos[2(\delta+1)]^{1/2}z & \delta \text{ odd} \end{cases} \tag{38}$$

for small z. We are assuming that plane wave radiation starts at $z = 0$ and then propagates into the material media. The normalization constants are then given by

$$C_\delta = \frac{(\delta/2)!}{\delta!}(-1)^{\delta/2} \qquad \delta \text{ even} \tag{39}$$

$$C_{\delta-1} = -(-1)^{\delta/2}\frac{(\delta/2)!}{\delta!}\sqrt{(2\delta)} \qquad \delta \text{ even} \tag{40}$$

$$D_\delta = \frac{(-1)^{\delta/2}[2(\delta+1)]^{1/2}}{2^\delta(\delta/2)!} \qquad \delta \text{ even} \tag{41}$$

$$D_{\delta+1} = \frac{-(-1)^{\delta/2}}{2^\delta(\delta/2)!} \qquad \delta \text{ even.} \tag{42}$$

The complete normalized solution of the field equations (20) is then

$$h_{jk}(z) = \mathrm{Re}\, e_{jk}\, \mathrm{e}^{\mathrm{i}\omega t}(C_\delta\, \mathrm{e}^{-z^2}H_\delta(z) + \mathrm{i}D_\delta\, \mathrm{e}^{-z^2}P_\delta(z)) + \mathrm{Re}\, e^*_{jj}\, \mathrm{e}^{\mathrm{i}\omega t}(C_\delta\, \mathrm{e}^{-z^2}H_\delta(z) - \mathrm{i}D_\delta\, \mathrm{e}^{-z^2}P_\delta(z)), \tag{43}$$

where h_{jk} is given as a function of the dimensionless distance of propagation into the medium z and as a function of frequency (through ω and δ).

We are now interested in the behaviour of our solution (43) for high frequencies. In particular we want to see if some sort of 'plasma frequency' seems to be present above which (43) behaves like a plane wave and below which significant absorption of the incident radiation occurs. We need H_δ and P_δ for very large δ. Unfortunately H_δ is tabulated only for small δ by Russel (1933) and P_δ is not tabulated at all. We calculated these quantities on the computer using the recursion relations and initial values of H_0, H_1, P_0 and P_1. The results were verified by several methods and shown to be accurate even for large δ. We actually calculated

$$S_\delta = \frac{1}{2^{\delta/2}\sqrt{(\delta!)}}H_\delta \tag{44}$$

which satisfies

$$S_{\delta+1}\sqrt{(\delta+1)} = zS_\delta\sqrt{2} - S_{\delta-1}\sqrt{\delta} \tag{45}$$

rather than H_δ since H_δ or P_δ becomes very large and S_δ does not.

For calculations we considered a series of stars *each of one solar mass* but with different mean densities. Densities in multiples of ten were chosen from $\rho = 1$ to 10^{14} g cm^{-3}. We calculated the solution (43) for z corresponding to propagation through a distance corresponding to the radius of the star for each ρ. Thus we calculated how much radiation would emerge from the surface of the star if a one-dimensional plane wave were emitted at the centre. This is only one of several ways in which numerical calculations based on our solution (43) could be presented. We chose this rather unphysical case because we are more interested in the density and frequency dependence of the wave propagation than in detailed geometrical considerations. Our calculation is clearly an approximation for problems with gravitational radiation impinging on the star from the outside, for the propagation of the gravitational radiation generated by a vibrating star through that star, for a three-dimensional wave of radiation travelling outward through a shell of matter in a supernova explosion, etc. The interesting density and frequency dependence we find below for the emerging radiation should be generally valid. We do not mean to imply that stars really do emit plane gravitational waves from their centres.

We would now like to find a measure of the amount of gravitational radiation emerging from the star. The amplitude of the wave H is given by

$$H^2 = \tfrac{1}{2}\langle h^{ij}h_{ij}\rangle \tag{46}$$

where the brackets indicate an average over a region of space corresponding to several wavelengths. It is necessary to do this because gravitational energy cannot be localized over distances less than a wavelength of the radiation according to Isaacson (1968). Putting (43) into (46) and discarding terms which go as $e^{\pm 2i\omega t}$ gives

$$H^2(z) = e^{jk}e^*_{jk}[(C_\delta \, e^{-z^2}H_\delta(z))^2 + (D_\delta \, e^{-z^2}P_\delta(z))^2]. \tag{47}$$

At $z = 0$ we have

$$H^2(z = 0) = e^{jk}e^*_{jk} \tag{48}$$

for the initial plane wave, since the z-dependent factor in (47) becomes

$$\cos^2[2(\delta+1)]^{1/2}z + \sin^2[2(\delta+1)]^{1/2}z = 1$$

as z approaches zero.

Another measure of the amount of radiation emerging from the star is the energy flux in the X direction,

$$t_{01} = \frac{c^5}{8\pi G}R_{01}{}^{(2)}, \tag{49}$$

where $R_{01}{}^{(2)}$ is the second-order part of the curvature tensor R_{01}. We have from Weinberg (1972)

$$R_{01}{}^{(2)} = -\frac{3}{4}\frac{i\omega}{c}h^{ij}h'_{ij} + \frac{i\omega}{2c}h^{i1}h'_{i1} \tag{50}$$

where we have used the TT coordinate conditions to greatly simplify this. A prime denotes differentiation with respect to X. If we limit ourselves again to the one-dimensional problem and assume that $h_{\mu\nu}$ is a function of X and t only, we obtain

$$t_{01} = \frac{-\tfrac{3}{4}i\omega h^{ij}h'_{ij}c^4}{8\pi G}. \tag{51}$$

If we again average over several wavelengths and divide by the flux at $X = 0$ we have

$$\frac{\langle t_{01}(z)\rangle}{\langle t_{01}(z=0)\rangle} = \left(\frac{2\delta}{2(\delta+1)}\right)^{1/2} D_\delta \, e^{-z^2}P_\delta(z)C_{\delta-1} \, e^{-z^2}H_{\delta-1}(z) + C_\delta \, e^{-z^2}H_\delta(z)D_{\delta-1} \, e^{-z^2}P_{\delta-1}(z) \tag{52}$$

where we used (32) and $z \equiv X\sqrt{(I/2)}$ to calculate h'_{ij}.

For the one-dimensional waves we are considering, neither the amplitude of the waves (47) nor their energy flux in the X direction (52) is as significant a measure of absorption as their momentum in the X direction, K_X. For a plane wave

$$\langle h^{ij}h'_{ij}\rangle \propto C^{ij}C^*_{ij}K_X \tag{53}$$

and

$$H^2 = \tfrac{1}{2}\langle h^{ij}h_{ij}\rangle = C^{ij}C^*_{ij}. \tag{54}$$

Then

$$\frac{K_X(X)}{K_X(X=0)} = \frac{\langle t_{01}(X)\rangle}{\langle t_{01}(X=0)\rangle} \frac{H^2(X=0)}{H^2(X)} \tag{55}$$

where we used (51), (53) and (54). Let us define the quantity in (55) for more general waves as the transmission coefficient $\mathscr{T}$. This should be a sensitive measure of the amount of momentum and hence energy absorbed as the wave propagates through the medium. We note that $\mathscr{T}$ is gauge invariant since Weinberg (1972) has shown that $\langle t_{\mu\nu}\rangle$ is gauge invariant and since the gauge invariance of $\langle t_{01}\rangle \propto (H^2)'$ assures the gauge invariance of H^2 in our case. Thus even though we have worked in the TT gauge throughout, our results in terms of $\mathscr{T}$ will be gauge invariant. We can calculate $\mathscr{T}$ for our waves readily from (47), (48) and (52). We expect $\mathscr{T} \leqslant 1$ for all frequencies and propagation distances z. Also from our original differential equation we expect plane wave type solutions for high frequencies (large $\delta \equiv J/I-1$ in (28)) and expect $\mathscr{T} \to 1$ for high frequencies.

The results for $\mathscr{T}$ as a function of frequency are shown for $\rho = 10^{10}, 10^{12}, 10^{13}$ and 10^{14} g cm^{-3} in figure 1. The mass of the star is fixed at $1M_\odot$ and we are calculating the fraction of the K_X which emerges from the star relative to the initial K_X at $X = 0$. It should be noted that our solution (43) is valid only for integer δ and is not valid for intermediate values of the frequency. One interesting value of $\mathscr{T}$ is that for $\omega \to 0$ where

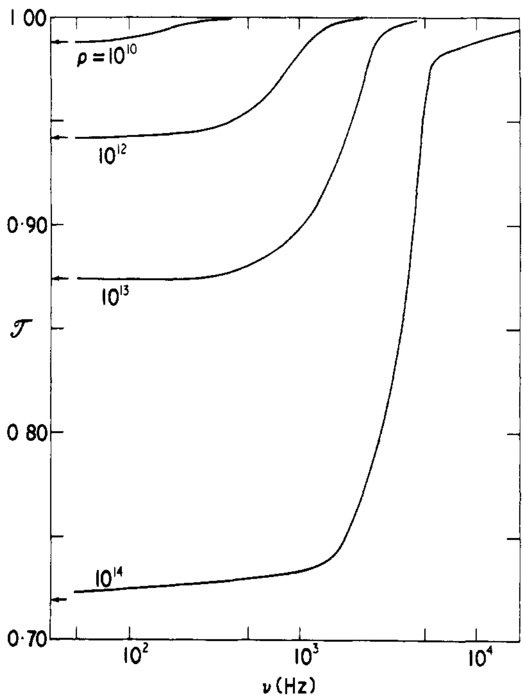

Figure 1. The transmission coefficient $\mathscr{T}$ is plotted as a function of frequency for various average densities ρ. The total mass of the star is fixed at $1M_\odot$. $\mathscr{T}$ gives the fraction of the initial momentum in the wave which successfully traverses a distance corresponding to the radius of the star. Small arrows on the left indicate $\mathscr{T}$ at $\nu = 0$ where the absorption is a maximum.

the absorption is maximum. This corresponds to $\delta = -\frac{2}{3}$, giving

$$Q_1(z) \simeq 1+\tfrac{2}{3}z^2+\tfrac{8}{27}z^4+\tfrac{112}{1215}z^6+\dots \quad (56)$$

The other solution to (29) is given by (34) and is

$$Q_2(z) \simeq \sqrt{(\tfrac{2}{3})}z(1+\tfrac{5}{9}z^2+\tfrac{11}{54}z^4+\dots) \quad (57)$$

where we have normalized as in (37) and (38). (56) and (57) can then be used to calculate the values of $\mathscr{T}$ at $\omega = 0$ as shown in figure 1.

3. Conclusions

An examination of figure 1 leads to several conclusions.

(i) $\mathscr{T} \rightarrow 1$ and our solutions become plane waves for high frequencies.

(ii) High-density stars absorb much more gravitational radiation than low-density stars of the same total mass.

(iii) The maximum absorption occurs at low frequencies and is 1 %, 6 %, 13 % and 28 % for $\rho = 10^{10}$, 10^{12}, 10^{13} and 10^{14} respectively. The absorption is insignificant for stars with $1M_{\odot}$ and $\rho < 10^{10}$ and is not shown in figure 1.

(iv) The transmission coefficient $\mathscr{T}$ shows a very marked step-like behaviour when plotted as a function of frequency. The step is much more pronounced at $\rho = 10^{14}$ than at $\rho = 10^{10}$. (The reader is cautioned that the frequency scale is logarithmic in figure 1. The sharp step persists in a linear plot but loses much of its low-frequency 'tail'.) This behaviour is highly reminiscent of the behaviour of electromagnetic waves traversing a plasma. There the transmission is one above the plasma frequency ω_p and exponential absorption occurs below ω_p. From figure 1 the critical frequencies above which our solutions are essentially plane waves are approximately 240, 1700, 4100 and 6700 Hz for $\rho = 10^{10}$, 10^{12}, 10^{13} and $10^{14}\,\mathrm{g\,cm^{-3}}$ respectively. In terms of δ these critical frequencies occur approximately at $\delta = 130$, 70, 40 and 10 respectively. Empirically we have then

$$\delta_{\text{crit}} \simeq 10+30\,\lg(10^{14}\,\mathrm{g\,cm^{-3}}/\rho) \quad (58)$$

at least in this density range. Now

$$\delta \sim \tfrac{1}{3}[\omega^2R^2/2-2] \simeq \tfrac{1}{3}\omega^2R^2/c^2.$$

Thus we can write the critical 'plasma frequency for gravitational radiation' as

$$\nu_{\text{crit}} \equiv \frac{\omega_{\text{crit}}}{2\pi} \simeq \frac{c\sqrt{30}}{2\pi R}[1+3\,\lg(10^{14}/\rho)]^{1/2}. \quad (59)$$

From (5) then we have

$$\nu_{\text{crit}} \simeq \sqrt{(20G\rho/\pi)}[1+3\,\lg(10^{14}/\rho)]^{1/2}. \quad (60)$$

In the electromagnetic case $\omega_p \propto \sqrt{N_e}$ and here $\omega_{\text{crit}} \mathrel{\tilde{\propto}} \sqrt{\rho}$ with a further weak ρ dependence from the log term in (59). In the electromagnetic case the transition from exponential attenuation to total transmission occurs very sharply at $\omega = \omega_p$. In the gravitational case the transition is slower though still rather sharp from figure 1. It should be noted that in figure 1 the critical wavelength is of the same order of magnitude as the radius of the star and this certainly influences the maximum attenuation obtained. Because of this we include figure 2 which shows what happens if we do not 'run out of star' at a particular density.

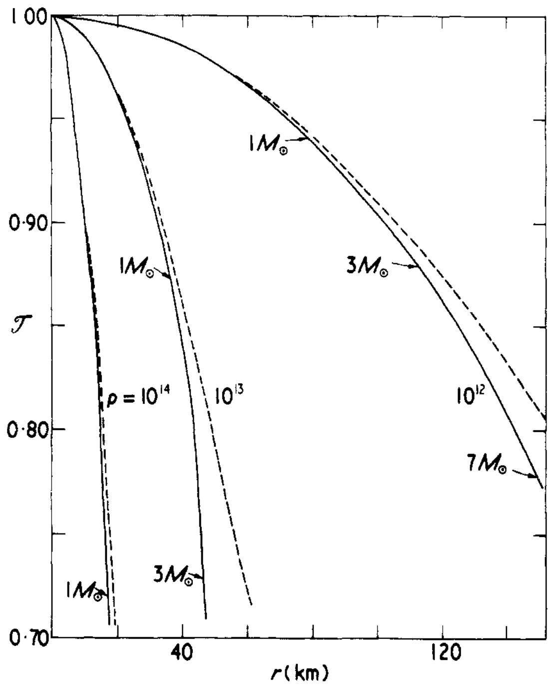

Figure 2. The full lines give the transmission coefficient $\mathscr{T}$ as a function of distance of propagation for zero frequency and for the indicated densities. Also indicated is the mass of the star in solar masses corresponding to that radius (or distance of propagation) and mean density. We are considering now a series of stars with different total mass or equivalently looking at $\mathscr{T}(r)$ within a large star. The broken curves shown for comparison are plots of $\exp(-3r^2/2R^2) \simeq e^{-z^2}$.

(v) Below the 'plasma frequency' we get attenuation that goes approximately as e^{-z^2} where z is our dimensionless distance variable. This is shown in figure 2 where the transmission coefficient $\mathscr{T}$ is plotted as a function of distance for zero frequency. The curves will be similar for any frequency below the critical frequency. Now we are essentially fixing ρ and considering a series of stars with different total mass or equivalently looking at the transmission as a function of distance of propagation within a large star. The dotted curves in figure 2 correspond to $\exp(-3r^2/2R^2)$ where R^2 is given by (5). Since

$$z^2 \simeq \tfrac{3}{2}(r^2/R^2)(1+\tfrac{13}{8}r^2/R^2)$$

this is approximately a plot of e^{-z^2}. Since z is always less than one for objects of physical interest and ranges from 0·11126 to 0·5924 as ρ increases from 10^{10} to 10^{14} g cm^{-3}, the gravitational attenuation occurs more slowly as a function of distance than the corresponding electromagnetic attenuation. Nevertheless we see that the concept of a 'gravitational plasma frequency' can be applied usefully to gravitational radiation propagating in dense media.

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References

Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (eds) (Washington: National Bureau of Standards) pp 771–802

Isaacson R A 1968 *Phys. Rev.* **166** 1272

Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) p 946

Møller C 1952 *The Theory of Relativity* (Oxford: Oxford University Press) p 332

Press W H and Thorne K S 1972 *Ann. Rev. Astron. Astrophys.* **10** 335

Russel J B 1933 *J. Math. Phys.* **12** 291

Weber J 1969 *Phys. Rev. Lett.* **22** 1320

——1970a *Phys. Rev. Lett.* **24** 276

——1970b *Phys. Rev. Lett.* **25** 180

Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) pp 167, 584